$$[\alpha\beta\gamma]=\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}[\varepsilon_1\varepsilon_2\varepsilon_3] \qquad (45).$$

Hence if $[\varepsilon_1\varepsilon_2\varepsilon_3]$ is the volume of a cube each side of which is a linear unit, $[\alpha\beta\gamma]$ devotes the volume of the parallelopiped whose adjacent edges are α, β, γ, since, as is well known in analytic geometry, the determinant expresses the number of units of volume in the parallelopiped. In the *Ausdehnungslehre* attention must be paid to the *order* of the factors, *i. e.* to the order of generation. Thus (37) $[\alpha\beta]=-[\beta\alpha]$, and $[\alpha\beta\gamma]=-[\alpha\gamma\beta]$.

It is apparent from the preceding articles that the *Ausdehnungslehre* is especially well adapted for the investigation of propositions concerning the areas and volumes of rectilinear figures.

91. *To find the product of a posited point and vector.*

Let $p\rho$ and ε denote the point and vector. Following the areal interpretation given above this product should be a *line* whose length is ε and whose other dimension is the infinitesimal p, the whole fixed in position by the radius vector ρ.

Now $p(\rho+x\varepsilon)$ denotes any point on the line through $p\rho$. Then since

$$[p(\rho+x\varepsilon)\varepsilon]=[p\rho\varepsilon]+x[p\varepsilon\varepsilon]=[p\rho\varepsilon] \ (34),$$

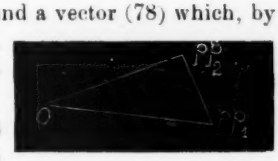
we see that the product of a posited point and a vector determine a line segment, *but this line segment may have any position on the vector through the given point.*

92. *To find the product of two posited points.*

Let $p\rho_1$ and $p\rho_2$ be two unit points. Then

$$[p\rho_1 . p\rho_2]=[p\rho_1(p\rho_2-p\rho_1)], \text{ since } [p\rho_1 . p\rho_1]=0. \ (34)$$

But $[p\rho_1(p\rho_2-p\rho_1)]$ is the product of a point and a vector (78) which, by 91, is the line from the extremity of ρ_1 to that of ρ_2. Thus, *The product of two posited points is the line joining the first to the second.*



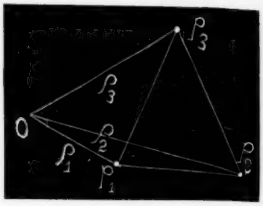
93. We have illustrated in the last article a principle which will be found to hold generally in the *Ausdehnungslehre*, viz., *That the product of posited quantities which have no common figure is some multiple of the connecting figure.*

94. *To find the product of three posited points.*

Let us use p_1, p_2, p_3 to denote the three unit points instead of $p\rho_1$, $p\rho_2$, $p\rho_3$ as heretofore. It will be understood when p is used to denote the posited point $p\rho$, that it stands for the complex quantity described in 76.

$$[p_1p_2p_3]=[p_1p_2 . p_3] \text{ (Rem. 13)} =[p_1p_2(p_3-p_1)], \text{ since } [p_1p_2p_1]=0 \text{ by 43.}$$

But $[p_1p_2]$ is the line from p_1 to p_2 (92), and by 88 the product of a line $[p_1p_2]$ and a vector p_3-p_1 (78) equals the parallelogram whose adjacent sides are $[p_1p_2]$ and $[p_1p_3]$.



Thus the product of three given posited points is twice the area of the triangle whose vertices are the three given points. (93)

Let $p_1+xp_2+yp_3$ denote any point in the plane of $[p_1p_2p_3]$.

Now $[p_1p_2p_3]$=twice area of triangle whose vertices are p_1, p_2, p_3; but we have

$$[(p_1+xp_2+yp_3)(p_2-p_1)(p_3-p_1)]=[p_1p_2p_3] \text{ (22, 43).}$$

This shows that *the value of the product remains the same whatever be the position of the triangle* $[p_1p_2p_3]$ *in the plane of these points.*

95. *To find the product of four posited points.*

Let p_1, p_2, p_3, p_4, represent four unit points. Then

$$[p_1p_2p_3p_4]=[p_1(p_2-p_1)(p_3-p_1)(p_4-p_1)] \text{ (43)}$$
$$=6\times\text{tetraedron whose vertices are } p_1,\ p_2,\ p_3,\ p_4, \text{ by 90.}$$

Let $p_1+xp_2+yp_3+zp_4$ be any point whatever. Then

$$[(p_1+xp_2+yp_3+zp_4)(p_2-p_1)(p_3-p_1)(p_4-p_1)]=[p_1p_2p_3p_4]. \text{ (22, 43)}$$

Hence the product of four points in solid space is the same no matter where located.

96. We have used the terms "line" and "line segment" to denote a quantity whose length and the line in which it must lie are given but not its position in that line. Similarly we will use the terms "plane" and "plane segment" (See 74) to denote the corresponding areal quantity described in 94. Grassmann's terms for them are respectively "*Linientheil*" and "*Flächentheil.*" For the quantity described in 95 he uses the term "*Körpertheil.*"

97. *To find the sum or difference of two lines or two planes.*

Let $[p_1p_2]$, $[p_1p_3]$ be the lines, $[p_1p_2p_3]$, $[p_1p_2p_4]$, the planes, and let $p_3+p_4=2p_s$, and $p_3-p_4=\varepsilon$. Then

$$[p_1p_2]\pm[p_1p_3]=[p_1(p_2\pm p_3)]=2[p_1p_s], \text{ or } [p_1\varepsilon], \text{ a line.} \quad (82, 91)$$

$$[p_1p_2p_3]\pm[p_1p_2p_4]=[p_1p_2(p_3\pm p_4)]=2[p_1p_2p_s], \text{ or } [p_1p_2\varepsilon], \text{ a plane.} \quad (94)$$

98. *To find the sum of the sides of a triangle.*

Let $p\rho_1$, $p\rho_2$, $p\rho_3$ represent the three vertices of a triangle. Then

$$[p\rho_1.p\rho_2]+[p\rho_2.p\rho_3]+[p\rho_3.p\rho_1]=[(p\rho_2-p\rho_1)(p\rho_3-p\rho_1)], \text{ since } [p\rho_1p\rho_2]=0.$$

Thus the sum of the three sides of a triangle equals the product of the vectors pp_2-pp_1 and pp_3-pp_1. This product differs from the expression for the area of the triangle (94) *by the absence of the first factor* pp_1. An interpretation of the expression given above for the sum of the sides which makes it equal to the area of the triangle may be had by thinking of pp as a *generative* product of p and p. Using the period to denote generative multiplication, we have

$$p.pp_1.pp_2=OAB\ ;\ p.pp_2.pp_3=OBC\ ;\ p.pp_3.pp_1=-OCA.$$

$$\text{Thus, } OAB+OBC-OCA=p.pp_1.p.pp_2+p.pp_3.pp_2+p.pp_2.pp_3=ABC.$$

Remark.—By Grassmann's formulas the sum of the sides of a triangle equals its area, for he treats a point as that which has position only, and considers that the product of two vectors alone equals the area described in 88. The use of the definition of a point given in 76 has the effect of making some of the theorems of this chapter depart in certain respects from Grassmann's. The writer thinks however that regarding a line as generated by a point in motion agrees well with Grassmann's conception of "generative" multiplication.

99. *If* $p_1, p_2, p_3, p_4, p_5, p_6$ *are six points,* $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ *are four vectors and* x *is any scalar, by the preceding articles we have the following conditions:*

(1) $p_1=xp_2$ is the condition that points p_1 and p_2 coincide.

(2) $[p_1p_2]=x[p_3p_4]$, is the condition that the (unlimited) lines $[p_1p_2]$ and $[p_3p_4]$ coincide.

(3) $[p_1p_2p_3]=x[p_4p_5p_6]$ is the condition that the (unbounded) planes $[p_1p_2p_3]$ and $[p_4p_5p_6]$ coincide.

(4) $\varepsilon_1=x\varepsilon_2$ is the condition that the vectors ε_1 and ε_2 are parallel.

(5) $[\varepsilon_1\varepsilon_2]=x[\varepsilon_3\varepsilon_4]$ is the condition that the planes of $[\varepsilon_1\varepsilon_2]$ and $[\varepsilon_3\varepsilon_4]$ are parallel.

100. A point is regarded as a space of the first order; an unlimited line as a space of the second order; an unbounded plane as a space of the third order; and solid space as a space of the fourth order. See 17 and 85. Since a vector may be regarded as a point at infinity, a vector also may be regarded as a quantity of the first order. See Chapter I.

101. Relative Products. A *Planimetric* product is a relative product whose factors are in a plane, or space of the third order. A Stereometric product is one whose factors are in a space of the fourth order (100).

102. *To find the planimetric product of two line segments* p_1p_2 and p_1p_3. We have

$$[p_1p_2.p_1p_3]=[p_1p_2p_3]p_1. \quad (67)$$

Here $[p_1p_2p_3]$ is a scalar (101, 61). Thus the product is the point of intersection multiplied by a scalar.

103. *To find the planimetric product of two parallel line segments* $[p_1p_2]$ *and* $[p_3p_4]$.

We have, by hypothesis, $p_3-p_4=x(p_1-p_2)$ (99, (4)). Then

$$[p_1p_2\cdot p_3p_4]=[(p_1-p_2)p_2\cdot(p_3-p_4)p_4] \text{ (49)}=x[(p_1-p_2)p_2\cdot(p_1-p_2)p_4] \text{ (Hyp.)}$$

$$=x[(p_1-p_2)p_2p_4](p_1-p_2) \text{ (67)}=[(p_3-p_4)p_2p_4](p_1-p_2) \text{ (Hyp.)}$$

$$=[p_3p_2p_4](p_1-p_2) \text{ (49)}=[p_2p_3p_4](p_2-p_1). \text{ (38)}$$

Hence the product is the point at infinity (the vector p_2-p_1) which is the intersection of the two lines multiplied by the scalar $[p_2p_3p_4]$.

104. The last two articles illustrate a principle of general application in the *Ausdehnungslehre*, viz., *That the relative product of posited quantities which have a common figure is that common figure multiplied by a scalar.* See 93.

105. *To find the planimetric product of two lines and a posited point.*

Let $[p_1p_2]$ and $[p_1p_3]$ be the lines and p the point. Then

$$[p_1p_2\cdot p_1p_3\cdot p]=[(p_1p_2p_3)p_1\cdot p] \text{ (13, Rem., 67)}=[p_1p_2p_3][p_1p]$$

since $[p_1p_2p_3]$ is a scalar. (101, 61)

106. *To find the planimetric product of the three line segments* $[p_1p_2]$, $[p_1p_3]$, $[p_4p_5]$.

We have $[p_1p_2\cdot p_1p_3\cdot p_4p_5]=[p_1p_2p_3][p_1\cdot p_4p_5]$ (67)$=[p_1p_2p_3][p_1p_4p_5]$ (55).

COROLLARY. *If the three line segments are the sides of a triangle we may write* $[p_2p_3]$ *instead of* $[p_4p_5]$. *Then the product is* $[p_1p_2p_3]^2$.

107. The following general principle is illustrated in the two preceding articles: *If at any time the product of factors combined in regular order from the left gives rise to a scalar or to a scalar times an extensive quantity, this scalar is to be regarded as a simple numerical factor, and the extensive quantity part of the product, if there is such, is to be combined with the remaining extensive factors, and so on.* Such products are described as *mixed*, *i. e.* as both progressive and regressive. (61)

108. *The stereometric products of a line and a point and of two lines are commutative; but that of a point and a plane is non-commutative.*

Let L_1 denote the line $[p_1p_2]$, L_2 the line $[p_3p_4]$, and P the plane $[p_2p_3p_4]$.

$$[L_1p_3]\equiv[p_1p_2\cdot p_3]=[p_1p_2p_3]=[p_3\cdot p_1p_2] \text{ (40, 55)}\equiv[p_3L_1];$$

$$[L_1L_2]\equiv[p_1p_2\cdot p_3p_4]=[p_1p_2p_3p_4] \text{ (55)}=[p_3p_4\cdot p_1p_2] \text{ (40, 55)}\equiv[L_1L_2];$$

$$[Pp_4]\equiv[p_1p_2p_3p_4]=-[p_4\cdot p_1p_2p_3] \text{ (40, 55)}\equiv-[p_4P].$$

109. *To find the stereometric product of two non-incident plane segments* $[p_1p_2p_3]$ and $[p_1p_2p_4]$.

We have $[p_1p_2p_3 \cdot p_1p_2p_4] = [p_1p_2p_3p_4][p_1p_2]$ (67). See 104.

110. *To find the stereometric product of three plane segments* $[p_1p_2p_3]$, $[p_1p_2p_4]$, $[p_1p_3p_4]$ *which intersect in* p_1.

$$[p_1p_2p_3 \cdot p_1p_2p_4 \cdot p_1p_3p_4] = [p_1p_2p_3p_4][p_1p_2 \cdot p_1p_3p_4]\ (67) = [p_1p_2p_3p_4]^2 p_1\ (67).$$

111. *To find the stereometric product of a plane segment* $[p_1p_2p_3]$ *and line segment* $[p_1p_4]$ *which do not lie in the same plane. Is the product commutative?*

We have $[p_1p_2p_3 \cdot p_1p_4] = [p_1p_2p_3p_4]p_1$ (67) (See 104).

Also, $[p_1p_4 \cdot p_1p_2p_3] = [p_1p_4p_2p_3]p_1\ (67) = [p_1p_2p_3p_4]p_1$ (40).

112. *The stereometric product of two line segments* $[p_1p_2]$ *and* $[p_3p_4]$ *equals zero when and only when they lie in the same plane* (55. 95) ; *the stereometric product of two quantities of the first, second, or third orders, but not both at the same time of the second orders, equals zero when and only when the quantities are incident, i. e. when one falls in the space of the other ; as two coincident points, two plane segments if the planes coincide, a point and a line- or a plane-segment if the point lies in the line or plane, a line segment and a plane segment if the line lies in the plane.*

113. *Algebraic Curves and Surfaces. The equation of a variable point* p *which lies in the same straight line as* $[p_1p_2]$ *is* $[p\ p_1p_2] = 0$. (112)

114. *The equation of a straight line* L *which passes through the intersection of* L_1 *and* L_2 *and lies in their plane is* $[L_1L_2L] = 0$ *where* $[L_1L_2L]$ *is a planimetric product* (See 106).

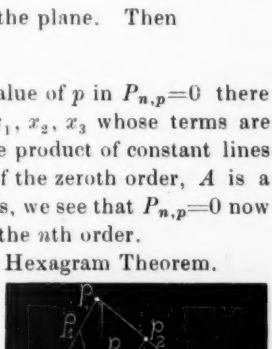
115. *If* $P_{n,p}$ *is a planimetric product of order zero which contains the variable point* p, n *times and besides only constant points and lines as factors, then* $P_{n,p} = 0$ *is the point equation of an algebraic curve of the* n*th order, that is to say, the point* p *moves in an algebraic curve of the* n*th order.*

Proof. Let p_1, p_2, p_3 be any three points in the plane. Then

$$p = x_1p_1 + x_2p_2 + x_3p_3$$

may be any point in the plane. Substituting this value of p in $P_{n,p} = 0$ there results a homogeneous equation of the nth degree in x_1, x_2, x_3 whose terms are all of the form $Ax^a{}_1x^b{}_2x^c{}_3$ where $a+b+c=n$. A is the product of constant lines and points, and since this product is by hypothesis of the zeroth order, A is a constant. Regarding x_1, x_2, x_3 as trilinear coördinates, we see that $P_{n,p} = 0$ now becomes an ordinary cartesian equation for a curve of the nth order.

116. As an example we give a proof of Pascal's Hexagram Theorem.



Let $p_1 \ldots p_5$ be five given points and p a variable point which moves so as to leave p'_2 on the line $[p'_1p'_3]$, p'_1, p'_2, p'_3 being defined by the following equations : $p'_1 = [pp_1 \cdot p_3p_5]$; $p'_2 = [p_2p_3 \cdot p_1p_4]$; $p'_3 = [pp_2 \cdot p_4p_5]$. $[(pp_1 \cdot p_3p_5)(p_2p_3 \cdot p_1p_4)(pp_2 \cdot p_4p_5)] = 0$ is the equation of a conic passing through the five given points. For it is of the second degree in p and is satisfied by putting p equal to any one of the five given points. By changing points into lines in the above we have Brianchon's Theorem.

[To be Continued.]

THE INTERNATIONAL CONGRESS OF MATHEMATICIANS.

By GEORGE BRUCE HALSTED.

On the sixth of August at the Palais des Congrès in the Paris Exposition, was held the opening session of the second International Congress of Mathematicians. The president, Poincaré, is regarded as the greatest of living mathematicians. Among the vice presidents in attendance were Gordan, Lindeloef, Lindemann, Mittag-Leffler.

Representing Japan was Fujisawa ; Spain sent Zoel de Galdeano ; the United States, Miss Scott.

The president of the section of Arithmetic and Algebra was Hilbert ; of Geometry was Darboux, of Bibliography and History was Prince Roland Bonaparte. Among the most interesting personalities present may be mentioned Dickstein of Warsaw, Gutzmer of Jena, Hagen of Washington, Laisant of Paris, Langel of Golfe Juan, Lemoine of Paris, Delury of Toronto, Padoa of Rome, Shroeder of Carlsruhe, Sintsof of Yekaterinoslav, Stringham of Berkeley, Tannery of Paris, Vasiliev of Kazan, Whitehead of Cambridge.

Of the many important papers presented two may be selected for their general interest and the enthusiasm with which they were received.

These are : The Mathamatics of the Old Japanese School by Fujisawa, and The Problems of Mathematics by Hilbert.

Among other matters of extraordinary importance, Fujisawa showed his astonished audience that the Japanese had independently discovered the zero and by a mysterious coincidence used for it a circular symbol as did the Hindus and as do we. He showed that the Japanese had rectified the circle with an accuracy far exceeding Archimedes and only paralleled in our modern developments of pure mathematics. He showed that the Japanese had recognized $\sqrt{-1}$ the square root of minus one as a number, as a new unit, a neomon, and thus had reached the basis for the theory of the complex numbers.

This paper is epoch-making for the history of mathematics.

Hilbert's beautiful paper on the Problems of Mathematics shows that when a science progresses continuously we may from the problems which actually occupy it judge of its ulterior development. The existence of precise problems has a capital importance both for the progress of mathematics and for the work of each investigator.

Whence come the problems of mathematics ? It is experience that in each domain puts before us the primary problems (*e. g.* duplication of the cube, quadrature of the circle, etc.) In the later development of the science it is the mind which by logical reasonings (combination, generalization, specialization) creates itself problems new and fertile (*e. g.* problems of prime numbers). We say that a problem is solved when starting from a finite number of assumptions furnished by the problem itself we demonstrate the justness of the solution

by a finite number of deductions. This mathematical rigor which we require does not necessitate complicated demonstrations; the most rigorous method is often the simplest and the easiest to comprehend.

The conceptions of arithmetic or those of analysis are not the only ones susceptible of rigorous treatment. Those of geometry and the physical sciences are equally so, provided that by means of a complete system of assumptions they are as well fixed as the conceptions of arithmetic.

When a problem presents serious difficulties, by what methods can we attack it?

First by *generalization*, in attacking the problem considered to a group of questions of the same order. (*E. g.* Introduction of ideal numbers into the theory of algebraic numbers; employment of complex paths in the theory of definite integrals).

Or else by *specialization*, in deepening the study of more simple analogous problems already solved.

The failure of attempts at the solution of a problem comes often from the problem being impossible to solve under the form given. Then we require a rigorous *demonstration of the impossibility*. (Parallel postulate, quadrature of the circle, algebraic solution of the equation of the fifth degree.)

We say that a conception exists from the mathematical point of view when the assumptions which define it are compatible, that is to say when a finite chain or system of logical deductions starting from these assumptions can never lead to a contradiction.

Mathematics in developing, far from losing its character of unique science, manifests it from day to day more clearly. Each real progress brings necessarily the discovery of methods more incisive and more simple, permitting to each geometer an access relatively facile to all the parts of our science.

The magnificent reception given by the President of France M. Loubet and his wife Madame Emilie Loubet in which the members of the Congress participated, was only surpassed in charm by the delightful entertainment given in our honor by Prince Roland Bonaparte.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ARITHMETIC.

130. Proposed by H. C. WHITAKER, M. E., Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pa.

Aow many balls 1 inch in diameter can be put in a cubical box 2 feet in the clear each way, putting in the maximum number?

I. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

$31\times\frac{1}{2}\sqrt{2}+1=22.9201.$

Hence, we can put in 32 layers ; 16 layers of 576 in each layer and 16 layers of 529 in each. There still remains space enough for one more layer.

$23\times\frac{1}{2}\sqrt{3}+1=20.918.$

Hence, in this layer, we can put 3 rows of 24 balls each and 24 rows of 24 and 23 alternately, or 636 in the whole layer.

$\therefore\ 16\times576=\ 9216$
$16\times529=\ 8464$
$1\times636=\ 636$

Total=18316

II. Solution by MARTIN H. SPINKS, Wilmington, Ohio.

Take the bottom layer and the rows in equilateral triangular form. The distance between the rows is .866 inch. The number of rows=1+(23÷.866) =1+26=27.

We then have 14 rows, 24 balls each, or 336 balls, and 13 rows, 23 balls each, or 299 balls each.

The bottom layer contains 336+299 or 635 balls. In the next layer we have 14 rows of 23 balls each or 322 balls, 13 rows of 24 balls each or 312 balls, in all 634 balls.

Distance between layers=.8162 inch.

Number of layers=1+(23÷.8162)=28+1=29.

Space left=24−(1+28×.8162)=.1464 inch.

$\therefore$ We have 15 layers, 635 balls each, or 9525 balls
and 14 layers, 634 balls each, or 8876 balls

Total=18401 balls.

NOTE. Excellent solutions of problem 129 were received from H. C. Whitaker, P. S. Berg, G. B. M. Zerr, Martin Spinks, J. Scheffer, and O. S. Westcott. Mr. Gruber also furnished a non-rythmical solution. We think that his poetical solution is sufficiently clear and accurate as to be easily understood. The results of the various contributors differ slightly from Mr. Gruber's and from each other.

ALGEBRA.

106. Proposed by ELMER SCHUYLER, B. Sc., Professor of German and Mathematics, Boys' High School, Reading, Pa.

$$\frac{x^2+x}{y^2+y}=a\ ;\quad \frac{x^2+y}{y^2+x}=b\ ;\ \text{find } x \text{ and } y.$$

I. Solution by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.; L. B. FILLMAN, Chester, Pa.; and the PROPOSER.

By composition and division we have at once

$$\frac{x^2+x+y^2+y}{x^2-y^2+x-y}=\frac{a+1}{a-1}\ldots(1),\quad \frac{x^2+y+y^2+x}{x^2-y^2+y-x}=\frac{b+1}{b-1}\ldots(2).$$

Dividing (1) by (2) and removing the factor $x-y$, we get at once,

$$\frac{x+y-1}{x+y+1}=\frac{(b-1)(a+1)}{(b+1)(a-1)}.$$

Whence $x+y=\frac{ab-1}{a-b}$, and $y=\frac{ab-1}{a-b}-x$.

Substitute this value of y in either of the original equations and a quadratic results, which of course can always be solved.

II. Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa., and J. SCHEFFER, A. M., Hagerstown, Md.

Let $x=vy$.

$$\therefore\ y=\frac{a-v}{v^2-a}=\frac{vb-1}{v^2-b}.$$

$$\therefore\ v^3(b+1)-v^2(a+1)-bv(a+1)+a(b+1)=0.$$

$$\therefore\ v=1,\ v=\frac{a-b\pm\sqrt{[4a(b+1)^2+(a-b)^2]}}{2(b+1)}.$$

$v=1$ makes $x=y=-1$ which is so when $a=b=0$.
Let the second value of $v=m$.

$$\therefore\ y=\frac{a-m}{m^2-a},\ x=\frac{am-m^2}{m^2-a}.$$

III. Solution by J. W. YOUNG, Fellow and Assistant in Mathematics, Ohio State University, Columbus, O.

$$\frac{x^2+x}{y^2+y}=a. \ldots(1),\ \frac{x^2+y}{y^2+x}=b. \ldots(2).$$

Let $x'=x+1$, $y'=y+1$.

Then (1) becomes $\frac{xx'}{yy'}=a$, or $xx'=ayy'. \ldots(3)$.

(2) may be written

$$x^2+x+y+by=by^2+by+x+bx, \text{ or } xx'+y(1+b)=byy'+x(1+b).$$

From (3) $\frac{(a-b)yy'}{1+b}=x-y=x'-y'$;

whence $x=y\left(y'\frac{a-b}{1+b}+1\right). \ldots(4),\ x'=y'\left(y\frac{a-b}{1+b}+1\right). \ldots(5).$

Multiplying (4) by (5),

$$xx'=yy'\left(y'\frac{a-b}{1+b}+1\right)\left(y\frac{a-b}{1+b}+1\right)=ayy'. \quad \text{From (3)}$$

$$\therefore yy'\left[\left(y'\frac{a-b}{1+b}+1\right)\left(y\frac{a-b}{1+b}+1\right)-a\right]=0. \quad . \quad .(6).$$

The first two factors $y=0$, $y'=0$, give the roots $y=0$, $y=-1$;
whence $x=0$, $x=-1$.

The remaining factor in (6) is clearly a quadratic for y, which yields the remaining two roots.

IV. Solution by W. F. SHAW, 1600 Sabine Street, Austin, Tex.

Clear of fractions and subtract

$$\begin{aligned} &x^2+x=ay^2+ay \\ &x^2+y=by^2+bx \\ \hline &x-y=(a-b)y^2+ay-bx \end{aligned}$$

Multiply first by b, second by a, and subtract.

$$\begin{aligned} &bx^2+bx=aby^2+aby \\ &ax^2+ay=aby^2+abx \\ \hline &(a-b)x^2+ay-bx=ab(x-y) \\ &(a-b)y^2+ay-bx=(x-y), \text{ above results.} \\ \hline &(a-b)(x^2-y^2)=(ab-1)(x-y) \\ &x+y=[(ab-1)/(a-b)] \end{aligned}$$

Combining this with one of the first equations the values of x and y are found to be

$$x=\frac{a+b+2ab\pm\sqrt{[4a(b^2+b+1)+(a+b)^2]}}{2(a-b)},$$

$$y=\frac{-(a+b+2)\mp\sqrt{[4a(b^2+b+1)+(a+b)^2]}}{2(a-b)}.$$

Combining the first two equations to eliminate one of the letters, a cubic results. Two roots having been found the third is quickly seen to be -1. The other set of values is then, $x=-1$, $y=-1$.

Solved in substantially the same manner by *J. M. BOORMAN*, Woodmere, N. Y.

GEOMETRY.

132. Proposed by ELMER SCHUYLER, B. Sc., Professor of German and Mathematics, Boys' High School, Reading, Pa.

To draw a circle to cut two given circles orthogonally.

I. Solution by VIRGINIA CRAIG, Senior Classical Student, Drury College, Springfield, Mo.

Let a and b be the radii of circles with centers P and Q, $a+b+c$ be the distance between P and Q, and the line joining P and Q cut the circles at E and F. Divide the line c in the ratio $c+2b : c+2a$, and let the point of division be O. Construct a semi-circle on OP as diameter, said semi-circle intersecting circle whose center is P at L. With O as center, OL as radius, describe a circle intersecting circle whose center is Q at M. This described circle will cut the two given circles orthogonally.

Proof. OL is perpendicular to PL.

$\therefore$ Circles with centers O and P intersect orthogonally.

$OE+OF=c$ and $OE : OF=c+2b : c+2a$.

$$\therefore OE=\frac{c(c+2b)}{2(a+b+c)},\quad OF=\frac{c(c+2a)}{2(a+b+c)}.$$

$$OL^2=OE(OE+2EP)=\frac{c(c+2b)}{2(a+b+c)}\left(\frac{c(c+2b)}{2(a+b+c)}+2a\right)$$

$$=\frac{c(c+2a)}{2(a+b+c)}\left(\frac{c(c+2a)}{2(a+b+c)}+2b\right)=OF(OF+2FQ).$$

Let OX be the tangent from O to circle with center Q. Then $OX^2=OF(OF+2FQ)$. But $OF(OF+2FQ)=OL^2=OM^2$.

$\therefore OX=OM$, and as M is on circumference of circle with center Q, OX and OM coincide.

$\therefore OM$ is tangent to circle with center Q and perpendicular to MQ.

$\therefore$ Circles with centers O and Q intersect orthogonally.

II. Solution by M. A. GRUBER, A. M., War Department, Washington, D. C.; J. W. YOUNG, Fellow and Assistant in Mathematics, Ohio State University, Columbus, O.; COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.; G. B. M. ZERR, A. M., Ph. D., Professor of Science and Mathematics, Chester High School, Chester, Pa.; M. E. GRABER, Heidelberg University, Tiffin, O.; and J. SCHEFFER, A. M., Hagerstown, Md.

Any circle with center on the radical axis and radius equal to the tangent will evidently satisfy the conditions of the problem. The problem, then, resolves itself into the construction of the radical axis of two circles.

If the circles intersect, the radical axis is the common chord. If they do not intersect, draw any other circle cutting the given circles and draw their common chords. Their point of intersection is the radical center of the three circles. The radical axis required must pass through this point and must be perpendicu-

lar to the line joining the centers of the given circles. This offers no difficulty and the problem is solved.

Also solved by *ELMER SCHUYLER, HALLET E. McCLINTOCK;* and *CHARLES C. CROSS.* Professor Young furnished a neat diagram to accompany his solution of Problem 131.

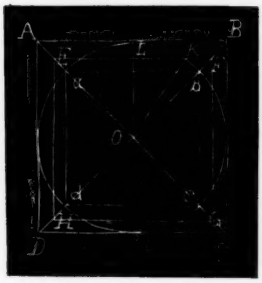
CALCULUS.

99. Proposed by **L. C. WALKER**, Associate Professor of Mathematics, Leland Stanford Jr. University, Palo Alto, Cal.

The axis of three equal right circular cylinders intersect at right angles. Find the volume of the solid common to all.

Solution by **G. B. M. ZERR, A. M., Ph. D.**, Professor of Mathematics and Science, Chester High School Chester, Pa.

Let $x^2+z^2=a^2$, $y^2+z^2=a^2$, $x^2+y^2=a^2$ be the equations of the cylinders. The projection of the intersection of the first and second on the plane xy is a square.



From $z=0$, to $z=\frac{1}{2}a\sqrt{2}$ the square decreases from $ABCD$ to $abcd$. The area of the projection common to the three cylinders between $z=0$ and $z=\frac{1}{2}a\sqrt{2}$ is eight times the area $LObKL$. From $z=\frac{1}{2}a\sqrt{2}$ to $z=a$ the common volume is the same as the volume common to the first and second.

Let $\angle LOK=\theta$. Area $EFGH=4xy=4(a^2-z^2)$.

Area $LOK=\frac{1}{2}a.OL\sin\theta=\frac{1}{2}ay\sin\theta$.

Area $KOb=\frac{1}{2}a^2(\frac{1}{4}\pi-\theta)$.

$\tan\theta=LK/OK=x/y=\sqrt{(a^2-y^2)}/y=z/\sqrt{(a^2-z^2)}$;

$\therefore \sin\theta=z/a$. $\therefore$ area $LOK=\frac{1}{2}z\sqrt{(a^2-z^2)}$, area $KOb=\frac{1}{2}a^2(\frac{1}{4}\pi-\sin^{-1}z/a)$.

$$\therefore V=8\int_0^{\frac{1}{2}a\sqrt{2}}[z\sqrt{(a^2-z^2)}+a^2(\tfrac{1}{4}\pi-\sin^{-1}z/a)]dz+8\int_{\frac{1}{2}a\sqrt{2}}^{a}(a^2-z^2)dz,$$

$$=8a^3(2-\sqrt{2}).$$

Also solved by *PROFS. ANDEREGG, SHERWOOD,* and *SCHMITT.*

100. Proposed by **B. F. FINKEL, A. M., M. Sc.**, Professor of Mathematics and Physics, Drury College, Springfield, Mo.

What is the volume bounded by the surface generated by the circumference of a circle whose diameter is the hypotenuse of a right-angled triangle whose base is b and altitude a, the plane of the circle being perpendicular to the plane of the triangle, the triangle and circle being rigidly connected, and the triangle revolving about its altitude a as an axis?

Solution by **F. ANDEREGG, A.M.**, Professor of Mathematics, Oberlin College, Oberlin, O., **H. C. WHITAKER, Ph. D.**, Professor of Mathematics, Manual Training School, Philadelphia, Pa.; **G. B. M. ZERR, A. M., Ph. D.**, Professor of Mathematics and Science, Chester High School, Chester, Pa.; and the **PROPOSER.**

Let $AB=a$, $OA=b$, $AC=\sqrt{(a^2+b^2)}$; the coördinates of P and point in

the given circumference, (x, y, z); and the coördinates of M, the projection of P on the xy plane, (x, y).

Then $DC=bx/a$; $BC=\sqrt{(a^2+b^2)}\,\frac{x}{a}$;

$CA=\frac{\sqrt{(a^2+b^2)}(a-x)}{a}$; and $PC=\sqrt{(BC.CA)}$

$$=\frac{\sqrt{[(a^2+b^2)(a-x)x}}{a}.$$

$$\therefore\ DP^2=\frac{b^2x^2+(a^2+b^2)(a-x)x}{a^2}.$$

As the circle with radius DP moves parallel to itself and with its center on AB, it generates the volume required.

$$\therefore\ V=\frac{\pi}{a^2}\int_0^a[b^2x^2+(a^2+b^2)(a-x)x]dx=\frac{\pi a}{6}(a^2+3b^2).$$

MECHANICS.

97. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

The side AB of the parallelogram $ABCD$ will be a principal axis at the point which divides the distance between the middle point and the foot of the perpendicular from the middle-point of the opposite side in the ratio 2 : 1. The principal moments of inertia about this point are $\frac{1}{3}mb^2\sin^2\beta$, $\frac{1}{36}m(3a^3+4b^2\cos^2\beta)$, where $\beta=\angle A$.

Solution by the PROPOSER.

Let $EH=c$, and let H be the origin, and lines through H parallel to EF, FB axes of coördinates.

$$\therefore\ \Sigma mxy=\rho\sin^2\beta\int_{-\frac{1}{2}a-c}^{\frac{1}{2}a-c}\int_0^b y(x+y\cos\beta)dxdy$$

$=\frac{1}{6}mb\sin\beta(2b\cos\beta-3c)=0$ if HB is a principal axis.

$\therefore\ c=\frac{2}{3}b\cos\beta$. But $FG=b\cos\beta$. $\therefore\ FH:HG=2:1$.

$$\Sigma my^2=\rho\sin^3\beta\int_{-\frac{1}{2}a-c}^{\frac{1}{2}a-c}\int_0^b y2dxdy=\tfrac{1}{3}mb^2\sin^2\beta.$$

$$\Sigma mx^2=\rho\sin\beta\int_{-\frac{1}{2}a-c}^{\frac{1}{2}a-c}\int_0^b(x+y\cos\beta)^2dxdy=\tfrac{1}{12}m(a^2+12c^2-12bc\cos\beta+4b^2\cos^2\beta)$$

$$=\tfrac{1}{36}m(3a^2+4b^2\cos^2\beta).$$

98. Proposed by WALTER H. DRANE, Graduate Student, Harvard University, Cambridge, Mass.

A spool, with light thread wound around, is placed upon a rough table so that the thread will emerge from beneath the spool. The thread is passed over a smooth pulley at end of table and a weight attached, the pulley being so adjusted that thread is parallel to surface of table. If friction between spool and table is sufficient to prevent slipping, determine motion of spool and weight. [From problems in Mechanica at Harvard University.]

I. Solution by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, O.

Let x and y be the horizontal and vertical parts of the string at any time t from the beginning of the motion, m the mass of the spool only, m' the mass attached to the thread, ϕ the angle of rotation of the spool in the time t, a and k the radius and radius of gyration of the spool, respectively, g the acceleration of gravity, T the tension in the thread, and F the friction.

The equation of motion of m' is $m'\frac{d^2y}{dt^2}=m'g-T$. . .(1).

For the linear and rotary motions of the spool,

$$m\frac{d^2x}{dt^2}=T-F. \ . .(2), \quad mk^2\frac{d^2\phi}{dt^2}=-Ta+Fa. \ . .(3).$$

Now $dx=ad\phi$. . .(4), and b being the initial length of free string, $x+y=b+a\phi$. . .(5).

From (4), $\frac{d^2\phi}{dt^2}=\frac{1}{a}\frac{d^2x}{dt^2}$. . .(6), and (3) is $mk^2\frac{d^2x}{dt^2}=-Ta^2+Fa^2$. . .(7).

From (2), $ma^2\frac{d^2x}{dt^2}=Ta^2-Fa^2$. . .(8).

(7) and (8) give $m(a^2+k^2)\frac{d^2x}{dt^2}=0$. . .(9), and $T=F$. . .(10).

From (5), $\frac{d^2x}{dt^2}+\frac{d^2y}{dt^2}=a\frac{d^2\phi}{dt^2}$. . .(11), which with (6) gives $\frac{d^2y}{dt}=0$..(12).

(9) gives $\frac{d^2x}{dt^2}=0$. . .(13), and (6) then gives $\frac{d^2\phi}{dt^2}=0$. . .(14).

Hence, since the system starts from rest, the last three equations show that there is no motion afterwards.

AVERAGE AND PROBABILITY.

89. Proposed by B. F. FINKEL, A.M., M.Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

An inch auger-hole is bored at random through a six-inch sphere. Find the average volume of the auger-hole.

Solution by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Let $x^2+y^2+z^2=R^2$ be the equation to the sphere.
Let $(x-a)^2+y^2=r^2$ be the equation to the auger.
Also let $\sqrt{[r^2-(x-a)^2]}=m$, $\sqrt{[R^2-r^2+a^2-2ax]}=n$.
From $a=0$, to $a=R-r$,

$$V=4\int_{a-r}^{a+r}\int_0^m \sqrt{[R^2-x^2-y^2]}\,dx\,dy=2\int_{a-r}^{a+r} mn\,dx$$

$$+2\int_{a-r}^{a+r}(R^2-x^2)\sin^{-1}\left(\frac{m}{\sqrt{(R^2-x^2)}}\right)dx=2\int_{a-r}^{a+r} mn\,dx$$

$$-\tfrac{2}{3}\int_{a-r}^{a+r}\frac{(3R^2x-x^3)(aR^2-R^2x+r^2x-a^2x+ax^2)\,dx}{mn(R^2-x^2)}$$

$$=2\int_{a-r}^{a+r} mn\,dx+\tfrac{2}{3}\int_{a-r}^{a+r}\frac{nx^2\,dx}{m}-\tfrac{2}{3}\int_{a-r}^{a+r}\frac{x(R^2-x^2)\,dx}{mn}+\tfrac{4}{3}R^2\int_{a-r}^{a+r}\frac{m\,dx}{n}$$

$$-\tfrac{4}{3}R^2\int_{a-r}^{a+r}\frac{x(a-x)\,dx}{mn}-\tfrac{2}{3}R^3\int_{a-r}^{a+r}\frac{m\,dx}{(R+x)n}-\tfrac{2}{3}R^3\int_{a-r}^{a+r}\frac{m\,dx}{(R-x)n}.$$

Let $x=a-r\cos 2\theta$, $R^2-(a-r)^2=b^2$, $4ar/b^2=e^2$, $R+a-r=d$.
$2r/d=c$, $R-a+r=h$, $-2r/h=c_1$, $\sqrt{[1-e^2\sin^2\theta]}=D$.

$$\therefore\ V=\tfrac{32}{3}br^2\int_0^{\frac{1}{2}\pi}D\sin^2\theta\cos^2\theta\,d\theta+\tfrac{16}{3}abr\int_0^{\frac{1}{2}\pi}D\sin^2\theta\,d\theta+\tfrac{4}{3}b(a-r)^2\int_0^{\frac{1}{2}\pi}D\,d\theta$$

$$+\frac{32ar^3}{3b}\int_0^{\frac{1}{2}\pi}\frac{\sin^6\theta\,d\theta}{D}+\frac{48ar^2(a-r)}{3b}\int_0^{\frac{1}{2}\pi}\frac{\sin^4\theta\,d\theta}{D}+\frac{8ar(4R^2-3b^2)}{3b}\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta\,d\theta}{D}$$

$$-\frac{4(a-r)(ab^2+2R^2r)}{3b}\int_0^{\frac{1}{2}\pi}\frac{d\theta}{D}-\frac{16R^3r^2}{3bd}\int_0^{\frac{1}{2}\pi}\frac{\sin^2\theta\cos^2\theta\,d\theta}{(1+c\sin^2\theta)D}$$

$$-\frac{16R^3r^2}{3bh}\int_0^{\frac{1}{2}\pi}\frac{\sin 2\theta\cos^2\theta\,d\theta}{(1+c_1\sin^2\theta)D}.$$

$$\therefore\ V=\tfrac{4}{3}\left[\frac{16br^2}{15e^4}(1-e^2+e^4)-\frac{4abr}{e^2}(1-2e^2)-\frac{8ar^3}{15be^6}(8+7e^2+8e^4)+b(a-r)^2\right.$$

$$-\frac{24ar^2(a-r)}{3be^4}(1+e^2)-\frac{2ar(4R^2-3b^2)}{be^2}-\frac{4R^3r^2}{bcde^2}-\frac{4R^3r^2}{bc_1he^2}\Big]E(e,\ \tfrac{1}{2}\pi)$$

$$+\tfrac{1}{3}\Big[-\frac{8br^2}{15e^4}(2-3e^2+e^4)+\frac{4abr}{e^2}(1-e^2)+\frac{8ar^3}{15be^6}(8+3e^2+4e^4)$$

$$+\frac{12ar^2(a-r)}{3be^4}(2+e^2)+\frac{2ar(4R^2-3b^2)}{be^2}$$

$$-\frac{(a-r)(ab^2+2R^2r)}{b}-\frac{4R^3r^2}{bc^2de^2}(ce^2+e^2-c)-$$

$$\frac{4R^3r^2}{bc_1^2he^2}(c_1e^2+e^2-c_1)\Big]F(e,\ \tfrac{1}{2}\pi)+\frac{16R^3r^2}{3b}\Big[\frac{c+1}{dc^2}\Pi(e,\ c,\ \tfrac{1}{2}\pi)+\frac{c_1+1}{hc_1^2}\Pi(e,c_1,\tfrac{1}{2}\pi)\Big]$$

$$\therefore\ V=AE(e,\ \tfrac{1}{2}\pi)+BF(e,\ \tfrac{1}{2}\pi)+C\Pi(e,\ c,\ \tfrac{1}{2}\pi)+C_1\Pi(e,\ c_1,\ \tfrac{1}{2}\pi),\ \text{suppose.}$$

Let $\dfrac{a^2+R^2-r^2}{2a}=x_1$; then from $a=R-r$ to $a=R+r$,

$$V_1=4\int_{x_1}^{R}\int_0^{\sqrt{(R^2-x^2)}}\sqrt{[R^2-x^2-y^2]}\,dxdy+4\int_{a-r}^{x_1}\int_0^{m}\sqrt{[R^2-x^2-y^2]}\,dxdy$$

$$=\pi\int_{x_1}^{R}\sqrt{[R^2-x^2]}\,dx+2\int_{a-r}^{x_1}mndx+2\int_{a-r}^{x_1}(R^2-x^2)\sin^{-1}\Big(\frac{m}{\sqrt{[R^2-x^2]}}\Big)dx.$$

$$\therefore V_1=\tfrac{1}{3}\pi[2R^3-(a-r)(2R^2+b^2)]+AE_0^{\theta_1}(e,\ \theta)+BF_0^{\theta_1}(e,\ \theta)+C\Pi_0^{\theta_1}(e,\ c,\ \theta)$$

$$+C_1\Pi_0^{\theta_1}(e,\ c_1,\ \theta),\ \text{where}\ \theta_1=\tfrac{1}{2}\cos^{-1}\Big(\frac{a^2+r^2-R^2}{2a}\Big).$$

$$\therefore\ \triangle=\frac{\int_0^{R-r}V da+\int_{R-r}^{R+r}V_1\,da}{\int_0^{R+r}da}.$$

A partial solution was given by *J. M. COLAW*.

MISCELLANEOUS.

83. Proposed by ALOIS F. KOVARIK, Instructor in Mathematics and Physics, Decorah Institute, Decorah, Iowa.

Segment's area of a circle whose radius is 5 inches is 28.56 square inches. Find the chord.

Solution by W.W. LANDIS, A.M., Professor of Mathematics and Astronomy, Dickinson College, Carlisle, Pa.; H. C. WHITAKER, Ph. D., Professor of Mathematics, Manual Training School, Philadelphia, Pa.; COOPER D. SCHMITT, A.M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.; G. B. M. ZERR, A.M.,Ph.D., Professor of Mathematics and Science, Chester High School, Chester, Pa.; J. SCHEFFER, A. M., Hagerstown, Md.

Let θ be the angle of the segment, c the chord.

$\therefore \frac{2}{3}5(\theta-\sin\theta)=28.56$, $c=10\sin\frac{1}{2}\theta$.

But $\theta-\sin\theta=2.2848$.

$\therefore \theta=155^\circ\ 4'$ nearly. $\therefore c=9.764$ inches.

84. Proposed by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.

Prove that $e^{\pi}+e^{-\pi}=2[1+2^2][1+(\frac{2}{3})^2][1+(\frac{2}{5})^2]\ldots$ad infinitum.

I. Solution by J. W. YOUNG, Fellow and Assistant in Mathematics, Ohio State University, Columbus, O.; J. O. MAHONEY, M. Sc., Professor of Mathematics, Central High School, Dallas, Tex.; HARRY S. VANDIVER, Bala, Montgomery Co., Pa.; M. E. GRABER, Heidelberg University, Tiffin, O.; ELMER SCHUYLER, B. Sc., Professor of Mathematics and German, Boys' High School, Reading, Pa.; J. SCHEFFER, A. M., Hagerstown, Md.; G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.; and PROPOSER.

From trigonometry we have the formula:

$$\cos\theta=\left(1-\frac{4\theta^2}{\pi^2}\right)\left(1-\frac{4\theta^2}{3^2\pi^2}\right)\left(1-\frac{4\theta^2}{5^2\pi^2}\right)\left(1-\frac{4\theta^2}{7^2\pi^2}\right)\ldots\ldots$$

In this replace θ by θi and we have

$$\cosh\theta=\left(1+\frac{4\theta^2}{\pi^2}\right)\left(1+\frac{4\theta^2}{3^2\pi^2}\right)\left(1+\frac{4\theta^2}{5^2\pi^2}\right)\left(1+\frac{4\theta^2}{7^2\pi^2}\right)\ldots\ldots$$

Now let $\theta=\pi$, whence

$$\cosh\pi=(1+4)\left(1+\frac{4}{3^2}\right)\left(1+\frac{4}{5^2}\right)\left(1+\frac{4}{7^2}\right)\ldots\ldots$$

$$=[1+2^2][1+(\tfrac{2}{3})^2][1+(\tfrac{2}{5})^2][1+(\tfrac{2}{7})^2].$$

But $\cosh\pi=\frac{1}{2}[e^{\pi}+e^{-\pi}]$, whence the result.

II. Solution by MISS MARY M. BLAINE, A.M., Teacher of Mathematics, High School, Springfield, Mo., and MISS ALICE MADELEINE McKELDON, Graduate Student, University of Pennsylvania, Philadelphia, Pa.

To prove that $e^{\pi}+e^{-\pi}=2[1+2^2][1+(\frac{2}{3})^2][1+(\frac{2}{5})^2]\ldots$

We have $(1)\cos x=1-\frac{x^2}{2\,!}+\frac{x^4}{4\,!}-\frac{x^6}{6\,!}+\ldots$

$$(2)\ \cos x=\left(1-\frac{4x^2}{\pi^2}\right)\left(1-\frac{4x^2}{3^2\pi^2}\right)\left(1-\frac{4x^2}{5^2\pi^2}\right)\ldots\ldots$$

[See Chrystal, Vol. II, page 329.]

Expanding this infinite product, and collecting,

$$\left(1-\frac{4x^2}{\pi^2}\right)\left(1-\frac{4x^2}{3^2\pi^2}\right).\ .\ .\ .\ .=1-\frac{4x^2}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\right]$$

$$+\frac{4^2x^4}{\pi^4}\left[\frac{1}{3^2}+\frac{1}{3^2\ 5^2}+-\right]-\frac{4^3x^6}{\pi^6}\left[\frac{1}{3^2\ 5^2}+-\right]+\ .\ .\ .$$

$$=1-c_1x^2+c_2x^4-c_3x^6+\ .\ .\ .(3).$$

Comparing (1), (2), and (3),

$$1-c_1x^2+c_2x^4-c_3x^6.\ .\ .\ .\ .=1-\frac{x^2}{2\ !}+\frac{x^4}{4\ !}-\frac{x^6}{6\ !}+\ .\ .\ .$$

$$\text{or}\left(1-\frac{4x^2}{\pi^3}\right)\left(1-\frac{4x^2}{3^2\pi^2}\right).\ .\ .=1-\frac{x^2}{2\ !}+\frac{x^4}{4\ !}.\ .\ .(4).$$

$$\text{Let } y^2=-x^2.\quad \left(1+\frac{4y^2}{\pi^2}\right)\left(1+\frac{4y^2}{3^2\pi^2}\right).\ .\ .=1+\frac{y^2}{2\ !}+\frac{y^4}{4\ !}+\frac{y^6}{6\ !}+\ .\ .\ .(5).$$

Now, adding this exponential series, e^y and e^{-y},

$$\frac{e^y+e^{-y}}{2}=1+\frac{y^2}{2\ !}+\frac{y^4}{4\ !}+\ .\ .\ .=\text{from (5)}\ \left(1+\frac{4y^2}{\pi}\right)\left(1+\frac{4y^2}{3^2\pi^2}\right).\ .\ .(6).$$

Substituting π for y in (6), we have,

$$e^{\pi}+e^{-\pi}=2[1+2^2][1+(\tfrac{2}{3})^2][1+(\tfrac{2}{5})^2].\ .\ .$$

PROBLEMS FOR SOLUTION.

ARITHMETIC.

132. Proposed by WILLIAM SYMMONDS, A.M., Professor of Mathematics, Santa Rosa College, Sebastopol, Cal.

A road 60 feet wide crosses a square acre of land. The west line of the road passes through the southwest corner of the land, while the east line of the former passes through the northeast corner of the latter. What fraction of the land is included in the road?

133. Proposed by COOPER D. SCHMITT, A. M., Professor of Mathematics, University of Tennessee, Knoxville, Tenn.

In Wentworth's Arithmetic he gives a formula $\frac{2}{4}\frac{1}{0}(d^2-2d)$ for calculating the number of board feet in a log 10 feet long, when d is the diameter in inches. How is this rule derived?

*** Solutions of these problems should be sent to B. F. Finkel not later than Nov. 10.

ALGEBRA.

121. Proposed by ELMER SCHUYLER, B. Sc., Professor of German and Mathematics, Boys' High School, Reading, Pa.

Solve $(x^5+y^5+z^5)^3+(x+y)^2=31$,
$(x^5+y^5+z^5)^3+(x+y+z)^3=729$,
$(x+y)^2+(x+y+z)^3=31$.

122. Proposed by JOSIAH H. DRUMMOND, LL. D., Portland, Me.

A man buys a five per cent. ten-year bond at such a price as enables him to spend annually three per cent. upon his investment and by continually investing the residue of the annual interest and its increase annually at four per cent., at the end of term upon payment of his bond has his original investment. What price per $100 does he pay for the bond?

*** Solutions of these problems should be sent to J. M. Colaw not later than Nov. 10.

GEOMETRY.

148. Proposed by DR. E. D. ROE, JR., Associate Professor of Mathematics in Syracuse University, Syracuse, N. Y.

The condition that two triangles, abc, xyz, are similar is

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ x & y & z \end{vmatrix} = 0,$$

and the condition that the triangle abc is equilateral is

$$\begin{vmatrix} a & b & 1 \\ b & c & 1 \\ c & a & 1 \end{vmatrix} = 0,$$

(Used in solving 130.)

149. Proposed by B. F. FINKEL, A. M., M. Sc., Professor of Mathematics and Physics, Drury College, Springfield, Mo.

Given a conic and two circumscribing triangles of the conic; prove that the six vertices of the triangles are con-conic.

150. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio University, Athens, O.

Find the equation to a sphere cutting orthogonally four given spheres.

*** Solutions of these problems should be sent to B. F. Finkel not later than Nov. 10.

CALCULUS.

112. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

A sphere of radius r is pierced by a cylinder radius $\frac{1}{2}r$ so that the cylinder just grazes the center of the sphere. Find volume removed; the lateral surface and the spherical surface removed.

113. Proposed by JOHN M. COLAW, A. M., Monterey, Va.

At what rate per unit of time are the roots of the equation $(x+px+q=0)$ changing, if $p=mq$ and q varies uniformly at the rate of 1/12 per unit of time, when $p=12$ and m remains constant?

*** Solutions of these problems should be sent to J. M. Colaw not later than Nov. 10.

MISCELLANEOUS.

93. Proposed by G. B. M. ZERR, A. M., Ph. D., Professor of Mathematics and Science, Chester High School, Chester, Pa.

Prove that $-(\sqrt{-1})^{\sqrt{-1}}=e^{(\sqrt{-1}-\frac{1}{2})\pi}$.

94. Proposed by J. SCHEFFER, A. M., Hagerstown, Md.

The wall of a house, if its plane were extended, would cut the horizon at an angle$=\beta^\circ$ south of the true east point. The latitude of the place being$=\phi$, and the declination of the sun$=\delta$. When will the sun cease to shine through a window in that wall?

95. Proposed by WILLIAM HOOVER, A. M., Ph. D., Professor of Mathematics and Astronomy, Ohio State University, Athens, O.

"I enjoy here," said Goethe, "both good days and good nights. Often before dawn I am already awake, and lie down by the open window, to enjoy the splendor of the three planets, which are at present to be seen together, and to refresh myself with the increasing brilliancy of the morning-red." This was written in the summer of 1828 near Weimar. See Goethe's "Conversations with Eckermann," Bohn's Library, 1898, page 323.

What three planets are referred to?

*** Solutions to these problems should be sent to J. M. Colaw not later than Nov. 10.

EDITORIALS.

Prof. Leslie L. Locke has been elected Instructor in Mathematics in the Michigan Agricultural College, Ingram County, Michigan.

Our valued contributor, Dr. E. D. Roe, Jr., has been elected Professor of Mathematics in the Syracuse University, Syracuse, N. Y.

Our good friend, Prof. George B. M. Zerr, has been called to the chair of Chemistry and Physics in The Temple College, Philadelphia, Pa.

Prof. Charles Scott Venable, LL. D., for the last five years Professor Emeritus of Mathematics at the University of Virginia, died on August 18 at his home in Charlottesville, Va. He was born in Prince Edward County, Va., on April 19, 1827. He received an academic education at the Hampden-Sidney College and the University of Virginia, being graduated at the former institution in 1842, and the latter in 1848. He continued his studies at Berlin in 1852, and at Bonn in 1854. Prior to 1861 he held professorships in the universities of South Carolina and Georgia, and in Hampden-Sidney College. In 1865 he was appointed professor of mathematics in the University of Virginia, and from 1870 to 1873 he was chairman of the Faculty. Professor Venable was the author of a series of text-books, including Arithmetic, Algebra, and Geometry.

BOOKS AND PERIODICALS.

The Theory and Practice of Interpolation, Including Mechanical Quadrature and other Important Problems Concerned with Tabular Values of Functions, with Requisite Tables. By Herbert L. Rice, M. A., Assistant in the office of the American Ephemeris and Professor of Astronomy in the Corcoran Scientific School, Washington, D. C. 4to cloth, 234 pages. Price, $3.50. Postage prepaid, $3.75. For sale by the author, United States Naval Observatory, Washington, D. C..

In the preparation of this treatise, the author has added to the technical literature of mathematics a work which gives a simple, practical, and comprehensive discussion of all that is useful concerning Differences, Interpolation, Tabular Differentiation, and Mechanical Quadrature, and has thereby rendered invaluable service to the practical computer. The work contains five chapters, an appendix, and eight tables. The first chapter treats of Differences; chapter II., of Interpolation; chapter III., of Derivatives of Tabular Functions; chapter IV., of Mechanical Quadrature; chapter V., of Miscellaneous Problems and Applications. The book is very beautifully printed with large type on heavy paper, and in every way makes a very good appearance. It is a work that not only every practical computer *must* have but every mathematician as well *should* have. B. F. F.

School Arithmetic, Primary Book. By J. M. Colaw, A. M., Associate Editor of the AMERICAN MATHEMATICAL MONTHLY, Monterey, Va., and J. K. Ellwood, A. M., Principal of the Colfax School, Pittsburg, Pa., author of Table Book and Test Problems in Elementary Mathematics. 8vo. cloth, 271 pages. Price, 35 cents. Richmond, Va.: B. F. Johnson Co.

This new votary for public favor in the line of arithmetics has many points of excellence to recommend it. The authors in the preparation of this work have certainly proved that they understand the fundamental principles of sound pedagogy. The first thirty pages are devoted to giving the child an idea of comparing objects of different size. For example, on the first page we find this: "Cut splints the same length as A, B, and C," these letters referring to the pictures of sticks. "Show me the longest," "show me the next longest," "show me the shortest," "show me the next shortest," etc. By such exercises, which every child can master, it is led to simple work in numbers. The book is well written, beautifully and tastefully printed and illustrated. Teachers desiring a good primary arithmetic will do well to adopt this work. B. F. F.

An Elementary Physics for Secondary Schools. By Charles Burton Thwing, Ph. D., Professor of Physics in Knox College, formerly instructor in the University of Wisconsin, author of Exercises in Physical Measurement, Part I., Principles, Part II., Laboratory Exercises. 8vo. cloth, 371 pages. Price, $1.20. Boston; Benj. H. Sanborn & Co., Publishers.

During the past five years wonderful discoveries have been made in physics. These, though great as they seem, probably very feebly foreshadow those yet to be made during the first decade of the coming Twentieth century. These great discoveries, following each other in rapid succession, have made text-books written six or seven years ago obsolete. To put the principles of these discoveries clearly before the public, has necessitated the revision of all old works on physics and the writing of numerous new ones. Professor Thwing's work belongs to the latter class, and the object of its publication is to give the average student of a secondary school a book scientific, accurate, and up-to-date. The only unfavorable criticism that may be offered is that some of the illustrations lack artistic effect. B. F. F.

Higher Algebra. By John F. Downey, M. A., C. E., Professor of Mathematics in the University of Minnesota, 8vo., cloth and leather back, 416 pages. Introduction price, $1.50. New York and Chicago: American Book Co.

This work is written for the use of students in technical schools, colleges and universities. Some of the characteristic features of this new book are: (1) The logical demonstrations, each theorem or general principle being followed by a concise logical demonstration; (2) Short processes are used instead of the longer ones in common use; (3) The subject of Maxima and Minima is discussed in a fuller and more systematic way than usual; (4) Differentiation of algebraic and logarithmic functions are introduced. Numerous exercises and problems are given in all the different subjects. B. F. F

Holden's Elementary Astronomy. By Edward S. Holden, M. A., Sc. D., LL. D., former Director of the Lick Observatory. With over 200 illustrations. xiv+446 pages, 12mo. Price, $1.20. New York: Henry Holt & Co. 1899.

This new volume in the "American Science Series" is addressed especially to pupils who are studying the subject for the first time. The author has endeavored to overcome the difficulties of this study by a very full and gradual treatment of its elements. Elementary instruction in observation is an important feature. The book is one of exceptional interest and merit. J. M. C.

History of English Literature. By F. V. N. Painter, A. M., D. D., Professor of Modern Languages and Literature in Roanoke College. 697 pages. Boston: Sibley & Ducker. 1899.

This work is an eminently practical text-book. It is characterized by judicious selection and wise omission. Unusual prominence has been given to the writers of the nineteenth century. The literary map, the list of books of reference, and that of "books worth reading" add much to the interest and value of the work. It is well printed and beautifully illustrated. J. M. C.

Plane Trigonometry with Tables. By Elmer A. Lyman, Michigan State Normal School, and Edwin C. Goddard, University of Michigan. Price, $1.00. Boston: Allyn & Bacon. 1899.

The book includes those portions of Plane Trigonometry studied in high school and college classes. The general character of the demonstrations, the early introduction of inverse functions, the extended practice in the use of logarithms, the use of oral work to aid in fixing formulæ in the mind, and frequent reviews, are some of the distinctive features. The trigonometric equation has received careful treatment, and in the solution of triangles the division into cases has been abandoned. J. M. C

Essentials of Plane and Solid Geometry. By Webster Wells, S. B., Professor of Mathematics in the Massachusetts Institute of Technology. 407 pages. Price, $1.25. Boston: D. C. Heath & Co. 1899.

In many of its features this work is similar to the author's Revised Plane and Solid Geometry, but important improvements have been introduced. The definitions and demonstrations are characterized by clearness, brevity, and accuracy. The book abounds in well-chosen and well-arranged exercises with excellent figures and suggestions. It ranks well with the very best books of its kind. J. M. C,

Mental Arithmetic. By Edward Weidenhamer, Ph. B. 173 pages. Cloth. Price, 35 cents. Harrisburg, Pa.: R. L. Myers &. Co. 1898, 1899.

This book begins with simple problems and proceeds by easy steps to those that are more difficult. The supply of problems is abundant, and in other respects this is a very satisfactory text. J. M. C.

The New Complete Arithmetic. By David M. Sensenig, M. S., and Robert F. Anderson, A. M., Instructors in Mathematics, State Normal School, West Chester, Pa. 427 pages. Price, 90 cents. Boston: Silver, Burdett & Co. 1900.

This book is designed to furnist to high schools, academies, and normal schools a complete treatise suitable for grades about to finish this branch of study. Under each subject examples of different types are solved in a manner calculated to direct attention to the logical steps involved. The very full treatment of business papers is a prominent feature. J. M. C.

The Elements of Arithmetic. By Ella M. Pierce, Supervisor of Schools, Providence, R. I. 149 pages. Price, 36 cents. Boston: Silver, Burdett & Co. 1900.

This book is intended for children of the third school year and covers the fundamental processes through numbers to one hundred. The lessons are simple and are well fitted to the age of the children for whom they are intended. J. M. C.

The Wooster Arithmetic—Grade I. By Lizzie E. Wooster. 112 pages. Cloth. Price, 25 cents. Topeka, Kansas: Crane & Co. 1899.

This book is intended for pupils in the first grade, and aims to do away with too much drill work upon the blackboard. The work advances by easy steps, and the number and variety of exercises are adequate. J. M. C.

The Elements of the Differential and Integral Calculus. By J. W. A. Young, Assistant Professor of Mathematical Pedagogy in the University of Chicago, and C. E. Linebarger, Instructor in Chemistry and Physics in the Lake View High School, Chicago. 410 pages. New York; D. Appleton & Co. 1900.

The present text is based closely upon the valuable German work of Professors Nernst and Schönflies which appeared in 1895. The fundamental principles and methods have been carefully treated in a manner that is in harmony with the more strict treatment possible in more extended treatises. The first chapter consists of an introduction to Analytic Geometry. Distinctive features are the exclusive use of the methods of limits, and the liberal use of illustrative examples from the natural sciences. The book has many points of excellence. J. M. C.

The Gospel According to Darwin. By Woods Hutchinson, A. M., M. D., of the University of Buffalo. 8vo. Paper cover. xii+241 pages. Price, 50 cents.

In this book the author has attempted to give merely a birds-eye-view of the influences affecting human hope and human happiness from the standpoint of that view of the universe and that attitude towards it which is best expressed by the term "Darwinism." In its pages are discussed in a very charming manner many of the themes that are most important to the human race. B. F. F.

We are indebted to Mr. C. M. Parker, Editor of *The School News and Practical Educator*, Taylorville, Ill., for copies of his valuable journal, containing a series of articles on Primary Number Work, by Prof. G. B. Longan.
J. M. C.

The following periodicals have been received since our last issue: *Journal de Mathématiques Élémentaires*, 15 Juillet 1900; *L'Intermédiaire des Mathématiciens*, Juillet 1900; *Notes and Queries*, September, 1900; *The Mathematical Gazette*, July, 1900; *The Educational Times*, August, 1900; *American Journal of*

Mathematics, July, 1900; *The Kansas University Quarterly*, *Proceedings of London Mathematical Society*, Vol. XXXI., *Periodico di Matematica*. J. M. C.

The American Monthly Review of Reviews. An International Illustrated Monthly Magazine, edited by Dr. Albert Shaw. Price, $2.50 per year in advance. Single number, 25 cents. The Review of Reviews Co., New York.

In the September *Review of Reviews* will be found a comprehensive treatment of the "Imperialism" issue, with particular reference to Mr. Bryan's Indianapolis speech. The editor's review of Mr. Bryan's Philippine propositions will be read with interest, alike by the adherents and the opponents of the Democratic candidate's policy. B. F. F.

The Literary Digest. A Weekly Compendium of the Contemporaneous Thought of the World. Price, $3.00 per year in advance. Single number, 10 cents. Funk & Wagnalls Co., Publishers, 30 Lafayette Place, New York.

A journal for the school and the home. In it is given a resumé of every important event in the civilized world. B. F. F.

The Cosmopolitan. An International Illustrated Monthly Magazine. Edited and published by John Brisben Walker. Price, $1.00 per year in advance. Single numbers, 10 cents. Irvington-on-the-Hudson.

Each number of this magazine is worth many times the yearly subscription price. In it culminates artistic excellence in magazine publications. B. F. F.

ERRATA.

Vol. VII, No. 4, page 104, second solution, the last line but two, for "37.7" read 31.5.

On line below the one just referred to, for "74.5" read 73.

On page 110, of the same number, for line 6, read $-\frac{8abr^2}{3c\sqrt{(a^2-r^2)}}F(e, \frac{1}{2}\pi)$.

On same page, instead of line 8, read $-\frac{8}{3}ab\sqrt{(a^2-r^2)}F(e, \frac{1}{2}\pi)$.

In No. 5, page 146, Problem 111, Calculus, for "of a hyperboloid or of a paraboloid," read, of a paraboloid or of a paraboloid.

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AN ELEMENTARY EXPOSITION OF GRASSMANN'S "AUSDEHNUNGSLEHRE," OR THEORY OF EXTENSION.

By JOS. V. COLLINS, Ph. D., Stevens Point, Wis.

[Continued from the August-September Number.]

CHAPTER VIII.

INNER PRODUCTS,—NORMAL SYSTEMS,—PROJECTION.

117. DEFINITION.—The Inner Product of two *units* of any order is the relative product of the first and complement of the second.

Thus the inner product of E and F is $[E \mid F]$.

Note.—Grassmann seems to have regarded the outer (52) and inner products as different in nature. But they both obey the laws of combinatory multiplication, the complement sign indicating a preliminary change to be made in the factor following it before it is combined with the other.

118. *The inner product of any two quantities is equal to the relative product of the first and complement of the second.*

PROOF.—Let $A=\alpha A_1+\ldots+\alpha_n A_n$, $B=\beta B_1+\ldots+\beta_n B_n$, where $A_1, \ldots,$ $B_1, \ldots,$ are units. Also for the moment let $\times$ signify the inner product.

$$\text{Then } [A \times B] \equiv [(\alpha_1 A_1+\ldots+\alpha_n B_n) \times (\beta_1 B_1+\ldots+\beta_n B_n)]$$

$$= \Sigma \alpha_r \beta_s [A_r \times B_s]. \quad (28)$$

Now since $A_1, \ldots, B_1, \ldots,$ are *units*, $[A_r \times B_s]=[A_r \mid B_s]$. (117)

Then $[A \times B]=\Sigma\alpha_r\beta_s[A_r \mid B_s]=\Sigma[\alpha_r A_r . \Sigma\beta_s \mid B_s]$ (28)

$$\equiv[A\Sigma\beta_s \mid B_s]=[A \mid \Sigma\beta_s B_s] \text{ (58) } \equiv[A \mid B].$$

119. *The inner product of two quantities of the same order is a number.* For, letting r denote the order of each factor, the complement of the second factor is of order n-r, and the product of the first factor which is of the order r and another which is of order n-r is of the nth order, *i. e.* is a pure number. (61)

COROLLARY.—*On account of the scalar value of the product, in this case* $[A \mid B]=[B \mid A]$.

120. *The inner product of two equal units is unity, while that of two different units of the same order is zero.*

Thus $[E_1 \mid E_1]=1$ (57), $[E_r \mid E_s]=0$. (43)

121. *If* $E_1, \ldots, E_n$ *are units of any order, but all of the same order, then*

$$[A \mid B]\equiv[(\alpha_1 E_1+\ldots+\alpha_n E_n) \mid (\beta_1 E_1+\ldots+\beta_n E_n)]$$

$$=\alpha_1\beta_1+\alpha_2\beta_2+\ldots+\alpha_n\beta_n. \text{ (120)}$$

122. *If* $B=A$ *in* 121, *we get what is called* the inner square of A, which is denoted by A^2; thus we have

$$A^2=\alpha_1^2+\alpha_2^2+\ldots\alpha_n^2.$$

123. NORMAL SYSTEMS.—DEFINITION.—The numerical value of an extensive quantity A is defined as the positive square root of the inner square of A. This definition reminds one of the modulus in complex numbers.

124. DEFINITION.—Two quantities (which do not equal zero) are said to be normal to each other if their inner product is zero. Two spaces are said to be every way (allseitig) normal to each other when each quantity in either space is normal to every quantity in the other space.

125. DEFINITION.—A normal system of the nth order is a set of n numerically equal quantities of the first order of which each is normal to every other. If at the same time n is the order of the space, then such quantities constitute a *perfect* normal system. The numerical value of these n quantities is at the same time the numerical value of the system. Every normal system whose numerical value is unity is called a *simple* system.

126. DEFINITION.—By Circular Alteration is meant that transformation of a system by which two quantities a and b of the system are transformed respectively into $xa+yb$ and $\pm(xb-ya)$, where $x^2+y^2=1$. The circular alteration is said to be positive or negative according as $+$ or $-$ is taken in the double sign.

127. *By circular alteration any normal system is transformed into another normal system having the same numerical value.*